Integralrechnung

Hauptsatz Analysis

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Ansatz & Korrektur   Ableitung

Muster: Verkettung und innere Ableitung

Bsp. $f(x) = x^{-3}\cdot \cos(x^{-2})$

$A(x) = \sin(x^{-2})$

$A'(x) = -2x^{-3}(\cos(x^{-2}))$

$K(x) = -\frac{1}{2}$

$F(x) = -\frac{1}{2}\sin(x^{-2})$

$= x^2\cdot e^{-2x^3}$

$= e^{-2x^3}$

$= -6x^2\cdot e^{-2x^3}$

$= -\frac{1}{6}$

$= -\frac{1}{6}\,e^{-2x^3}$

Partielle Integration (PI)

$$\int f'(x)\cdot g(x)\,dx \underset{PI}{=} f(x)\cdot g(x) - \int f(x)\cdot g'(x)\,dx$$

Bsp. $f(x) = x\cdot\sin(2x)$

Rollen

$f(x) = x \quad g'(x) = \sin(2x)$

$f'(x) = 1 \quad g(x) = -\frac{1}{2}\cos(2x)$

$\int x\cdot \sin(2x)\,dx$

$= -\frac{1}{2}x\cdot\cos(2x) + \frac{1}{2}\int\cos(2x)\,dx$

$= -\frac{1}{2}x\cdot\cos(2x) + \frac{1}{4}\sin(2x)$

Partialbruchzerlegung (PBZ)

$f(x) = \frac{Z(x)}{N(x)}$ grad $N(x)$ > grad $Z(x)$!

Bsp. $f(x) = \frac{-x^2+2x-17}{x^3-7x^2+11x-5}$

S1) $N(x) = (x-5)(x-1)^2$

S2) $\frac{Z(x)}{N(x)} = \frac{A}{x-5} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \quad | \cdot HN\ (x-5)(x-1)^2$

S3) $-x^2+2x-17 \equiv A(x-1)^2 + B(x-1)(x-5) + C(x-5)$

$= (A+B)x^2 + (-2A-6B+C)x + (A+5B-C)$

$$\begin{cases} A+B=-1 \\ -2A-6B+C=2 \\ A+5B-5C=-17 \end{cases} \qquad A=-2,\ B=1,\ C=4$$

S4) $\int\left[-\frac{2}{x-5} + \frac{1}{x-1} + \frac{4}{(x-1)^2}\right]dx$

$= -2\ln(|x-5|) + \ln(|x-1|) - 4(x-1)^{-1}$

Bsp. $f(x) = \frac{2x}{(x+1)(x-3)}$

S1) $N(x) = (x+1)(x-3)$

S2) $\frac{Z(x)}{N(x)} = \frac{A}{(x+1)} + \frac{B}{(x-3)} \quad | \cdot HN\ (x+1)(x-3)$

S3) $2x \equiv A(x-3) + B(x+1) = (A+B)x + (-3A+B)$

$$\begin{cases} A+B=2 \\ -3A+B=0 \end{cases} \qquad A=\frac{1}{2},\ B=\frac{3}{2}$$

$\int\left[\frac{1}{2}\,\frac{1}{x+1} + \frac{3}{2}\,\frac{1}{x-3}\right]dx = \frac{1}{2}\ln(|x+1|) + \frac{3}{2}\ln(|x-3|)$

Variablen-Transformations-Formel (VTF)

$$\int_{\varphi(a)}^{\varphi(b)} f(y)\,dy \overset{VTF}{=} \int_a^b f(\varphi(x))\cdot \varphi'(x)\,dx$$

Bsp. $f(y) = \sqrt{1-y^2}$

S1) $\varphi(a) = 0 : \sin(a) = 0,\ a = 0$
$\varphi(b) = 1 : \sin(b) = 1,\ b = \frac{\pi}{2}$

$$\int_0^1 \sqrt{1-y^2}\,dy = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2(x)}\cdot \cos(x)\,dx$$

S2) Ansatz wählen (Ansatz & Korrektur, PI, PBZ)
& Auflösen ...

Bogenlänge L(γ)

$$L(\gamma) = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

Bsp. $f(x) = \frac{1}{6}(x^3 + 3x^{-1})$ $\quad x_1 = 1,\ x_2 = 2$

1) $f'(x) = \frac{1}{6}(3x^2 - 3x^{-2}) = \frac{1}{2}(x^2 - x^{-2})$
2) $(f'(x))^2 = \frac{1}{4}(x^4 - 2 + x^{-4})$
3) $1 + (f'(x))^2 = \frac{1}{4}(x^4 + 2 + x^{-4}) = \frac{1}{4}(x^2 + x^{-2})^2$
4) $\sqrt{1+(f'(x))^2} = \frac{1}{2}(x^2 + x^{-2})$
5) $L(\gamma) = \frac{1}{2}\int_1^2 (x^2 + x^{-2})\,dx = \frac{1}{2}\left(\frac{x^3}{3} - x^{-1}\right)\Big|_1^2$

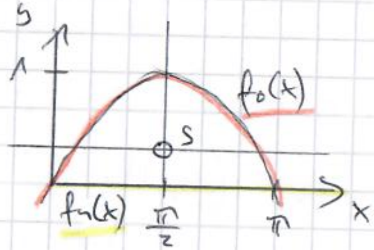
$= \frac{1}{2}\left[\frac{8}{3} - \frac{1}{2} - \left(\frac{1}{3} - 1\right)\right] = \underline{\frac{17}{12}}$

Flächenschwerpunkt S

$$\frac{1}{A}\int_a^b x\,(f_o(x) - f_u(x))\,dx =: x_S$$

$$\frac{1}{2A}\int_a^b (f_o^2(x) - f_u^2(x))\,dx =: y_S$$

Bsp. $f_u(x) = 0$
$f_o(x) = \sin(x)$



$A = \int_0^\pi \sin(x)\,dx = -\cos(x)\Big|_0^\pi$

$= -(\cos(\pi) - \cos(0)) = 2$

$x_S = \frac{1}{A}\int_0^\pi x\cdot\sin(x)\,dx \overset{PI}{=} \frac{1}{2}\left[-x\cos(x)\Big|_0^\pi + \int_0^\pi \cos(x)\,dx\right]$

$= \underline{\frac{\pi}{2}}$

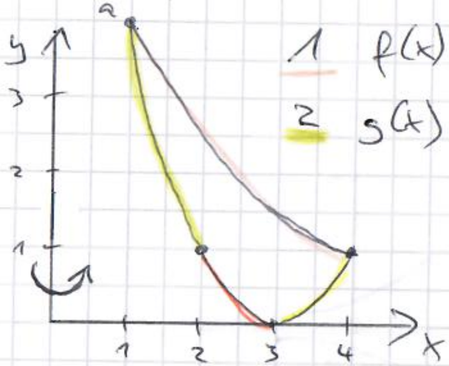
$y_S = \frac{1}{A} = \int_0^\pi \sin^2(x)\,dx = \frac{1}{4}\cdot\frac{\pi}{2} = \underline{\frac{\pi}{8}}$

Integrale ausrechnen: $\quad \Rightarrow S(x_S, y_S)$

- Ansatz & Korrektur
- Partielle Integration
- Partialbruchzerlegung
- Variablen-Transformations-Formel

Volumeninhalt

$$\pi \int_a^b f^2(x)\,dx = V$$



Bsp.

1 $f(x) = (x-3)^2$

2 $g(x) = \frac{4}{x}$

3 $3+\sqrt{y}$

4 $3-\sqrt{y}$

1 $x = \frac{4}{y}$ / $V_1 = \pi \int_1^4 \left(\frac{4}{y}\right)^2 dy = 12\pi$

2 $x = 3-\sqrt{y}$ / $V_2 = \pi \int_1^4 (3-\sqrt{y})^2 dy = \frac{13\pi}{2}$

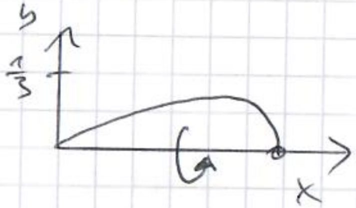
3 $x = 3+\sqrt{y}$ / $V_3 = \pi \int_0^1 (3+\sqrt{y})^2 dy = \frac{27\pi}{2}$

4 $x = 3-\sqrt{y}$ / $V_4 = \pi \int_0^1 (3-\sqrt{y})^2 dy = \frac{11\pi}{2}$

$$V = (V_1 - V_2) + (V_3 - V_4) = \underline{\frac{27}{2}\pi}$$

Mantelflächeninhalt

$$2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\,dx = M$$



Bsp.

$8y^2 = x^2(1-x^2)$

$y = f(x) = \sqrt{\frac{1}{8}x^2(1-x^2)}$

1) $f'(x) = \dfrac{\frac{1}{8}(2x-4x^3)}{2\sqrt{\frac{1}{8}x^2(1-x^2)}} = \dfrac{x(1-2x^2)}{8\sqrt{\frac{1}{8}x^2(1-x^2)}}$

2) $(f'(x))^2 = \dfrac{x^2(1-2x^2)^2}{8x^2(1-x^2)} = \dfrac{1}{8}\,\dfrac{(1-2x^2)^2}{1-x^2}$

3) $1+(f'(x))^2 = \dfrac{8-8x^2+1-4x^2+4x^4}{8(1-x^2)} = \dfrac{(3-2x^2)^2}{8(1-x^2)}$

4) $f(x)\cdot\sqrt{1+(f'(x))^2} = \frac{1}{8}x(3-2x^2)$

$$M = \frac{2\pi}{8}\int_0^1 (3x-2x^3)\,dx = \underline{\frac{\pi}{4}}$$

Bernoulli-Regel (B&H)

mit 3 Voraussetzungen gilt:

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

① $\lim\limits_{x\to x_0} f(x) = 0$ & $\lim\limits_{x\to x_0} g(x) = 0$

② $g(x) \neq 0$ & $g'(x) \neq 0$

③ $\lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)}$ muss existieren

Bsp. $\frac{x^2+x-6}{x^2-4}$ $\quad x_0 = 2$

$$\left.\begin{array}{ll} f(x) = x^2+x-6 \\ f'(x) = 2x+1 \end{array}\right\} \begin{array}{ll} g(x) = x^2-4 \\ g'(x) = 2x \end{array} \quad ①②$$

$$\frac{f'(x)}{g'(x)} = \frac{2x+1}{2x} \xrightarrow[\text{ans}]{x\to 2} = \underline{\frac{5}{4}} = \lim_{x\to x_0} \frac{f(x)}{g(x)}$$

③

Uneigentliches Integral

⇒ unbeschränkte Integral oder Integrations-Bereich

① Weg von Gefahr ② Grenzwert mit B&H

Bsp. $f(x) = \frac{1}{\sqrt{9-x^2}}$

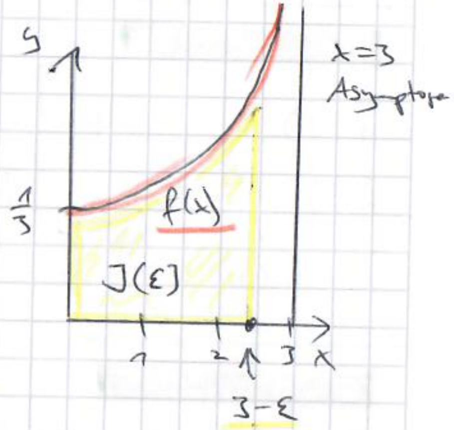
$$J(\varepsilon) = \int_0^{3-\varepsilon} \frac{1}{\sqrt{9-x^2}}\,dx$$

$$= \frac{1}{3}\int_0^{3-\varepsilon} \frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}}\,dx$$

$$= \arcsin\left(\frac{x}{3}\right)\Big|_0^{3-\varepsilon}$$

$$= \arcsin\left(1-\frac{\varepsilon}{3}\right) \xrightarrow[\text{ans}]{\varepsilon\to 0} \underline{\frac{\pi}{2}}$$

②



①

Bsp. $f(x) = \frac{1}{x^2+4}$

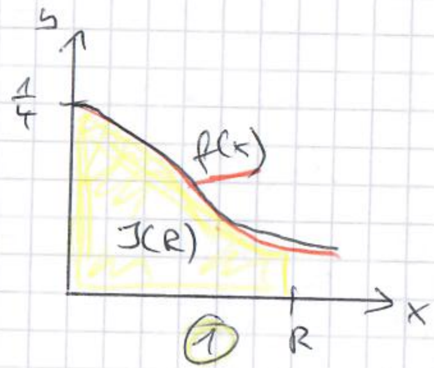
$$J(R) = \int_0^R \frac{1}{x^2+4}\,dx$$

$$= \frac{1}{4}\int_0^R \frac{1}{1+\left(\frac{x}{2}\right)^2}\,dx$$

$$= \frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big|_0^R$$

$$= \frac{1}{2}\arctan\left(\frac{R}{2}\right) \xrightarrow[\text{ans}]{R\to\infty} \underline{\frac{\pi}{4}}$$

②



①

Potenzreihen

Basics Konvergenz & Divergenz

Fakultät $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

Konvergenz ⇒ geht gegen eine Schranke

Divergenz ⇒ geht nicht gegen eine Schranke

Integral-Kriterium IK

Vorgehen in 4 Schritten

① Vermutung ⇒ konvergent oder divergent

② Orientierung ⇒ k mind. $\frac{1}{k^2}$ / d höchstens $\frac{1}{k}$

③ Strategie ⇒ k Majorante / d Majorante

④ Vergleichsfunktion und Summen skizzieren

Bsp. konvergente Majorante

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \boxed{\frac{1}{4} + \dots + \frac{1}{n^2}}$$

$$\leq 1 + \int_1^n x^{-2}\,dx = \leq 2$$

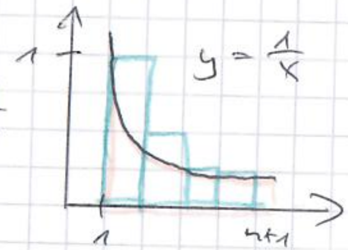
⇒ s_n wächst nach oben beschränkt

⇒ s_n wächst monoton

Bsp. divergente Majorante

$$s_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \dots + \frac{1}{n}$$

$$\geq \int_1^{n+1} \frac{1}{x} = \ln(n+1)$$



Quotenkriterium QK

$$s_n = \sum_{k=1}^{n} a_k$$

$$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \begin{cases} r < 1: & s_n \text{ konvergiert absolut} \\ r > 1: & s_n \text{ divergiert absolut} \\ r = 1: & \text{QK liefert kein Entscheid} \end{cases}$$

① $s_n = \sum_{k=1}^{n} a_k$ konvergiert, wenn $\sum_{k=1}^{n} |a_k|$ Grenzwert hat

② konvergiert $\sum_{k=1}^{n} |a_k|$, dann auch $\sum_{k=1}^{n} a_k$

Bsp. $s_n = \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \dots$

⇒ $a_k = \frac{k}{a^k}$ für $k = 1, 2, \dots$

$$\left| \frac{a_{k+1}}{a_k} \right| = \frac{k+1}{3^{k+1}} \cdot \frac{3^k}{k} = \frac{1}{3}\left(1 + \frac{1}{k}\right)$$

$$\xrightarrow[\text{GWS}]{k \to \infty} \underline{r = \frac{1}{3}} \quad s_n \text{ konvergiert absolut}$$

Vergleichskriterium VK

Vergleichsreihe suchen & vergleichen ...

Bsp. $s_n = \sum_{k=2}^{n} \frac{1}{k^3-1} \leq \sum_{k=2}^{n} \frac{1}{k^2}$

$\Rightarrow$ konvergiert

Bsp. $s_n = \sum_{k=1}^{n} \frac{k+2}{k(k+1)} \geq \sum_{k=1}^{n} \frac{1}{k}$

$\Rightarrow$ divergiert

Potenz-Reihe / Konvergenzbereich

$s_n(x) = \sum_{i=0}^{n} a_i(x) \quad x \in I$

offenes Intervall I, sodass $s_n(x)$ gegen einen Grenzwert $s(x)$ konvergiert für jedes $x \in I$

$\sum_{i \geq 0} a_i (x-x_0)^i \qquad i \geq 0$

Potenzreihe mit Entwicklungszentrum x_0

Bsp. $\frac{x-2}{1} + \frac{(x-2)^2}{2^2} + \frac{(x-2)^3}{3^2} + \frac{(x-2)^4}{4^2} + \dots$

$= \sum_{i \geq 0} \frac{(x-2)^i}{i^2} \qquad a_i(x) = \frac{(x-2)^i}{i^2} \quad a_{i+1}(x) = \frac{(x-2)^{i+1}}{(i+1)^2}$

$$\left| \frac{a_{i+1}(x)}{a_i(x)} \right| \underset{QK}{=} \left| \frac{(x-2)^{i+1}}{(i+1)^2} \cdot \frac{i^2}{(x-2)^i} \right| = \frac{1}{1+\frac{1}{i}} \cdot |x-2|$$

$\xrightarrow[\text{aus}]{i \to \infty} \quad r = |x-2| < 1 \Rightarrow I\,(1,3)$

Taylorpolynom

$$T_n f(x,a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} \cdot (x-a)^k$$

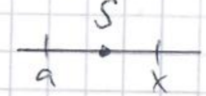
Beispiele

a) $f(x) = e^x$, $a=0$, $n=3$:
$T_3 f(x,a) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \quad \Rightarrow \sum_{k \geq 0} \frac{1}{k!} \cdot x^k$

b) $f(x) = \sin(x)$, $a=0$, $n=5$:
$T_5 f(x,a) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \quad \Rightarrow \sum_{k \geq 0} (-1)^k \cdot \frac{x^{(2k+1)}}{(2k+1)!}$

c) $f(x) = \cos(x)$, $a=0$, $n=4$:
$T_4 f(x,a) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \quad \Rightarrow \sum_{k \geq 0} (-1)^k \cdot \frac{x^{2k}}{(2k)!}$

Taylor + Restglied von Lagrange



$f(x) = T_n(x,a) + R_n f(x,a)$

$R_n f(x,a) = \frac{f^{(n+1)}(\xi)}{(n+1)!}$

für $R_n f(x,a) \xrightarrow{n\to\infty} 0 \Rightarrow \sum_{n \geq 0} \frac{f^{n}(a)}{n!} (x-a)^n$

(für Sinus, Cosinus & Exponentialfunktion geht das Restglied gegen 0.)

$e^{ix} = \cos(x) + i \sin(x)$ Eulerformel

$[\, i^2 = -1 \,]$

Geometrische Interpretation

ODE : $\begin{cases} y'(t) = f(t, y(t)) \\ y(0) = y_0 \end{cases}$: IC

f Vektorfeld (VF)
ODE Ordinary Differential Equation
IC Initial Condition

gesucht:

Funktion $y(t)$ mit IC, die ODE für alle Zeiten (t) aus einem offenen Intervall mit $0 \in I$.

autonom => hängt nicht von der Zeit ab

Bsp. $\begin{cases} y'(t) = \frac{3}{2}t^2 - \frac{1}{2}y(t) \\ y(0) = 1 \end{cases}$

$y(t) = 3(t^2 - 4t + 8) - 23e^{-\frac{1}{2}t}$

IC : $y(0) = 3 \cdot 8 - 23 = 1$ ✓

ODE : $y'(t) = 3(2t - 4) + \frac{23}{2}e^{-\frac{1}{2}t}$

$= \frac{3}{2}t^2 - \frac{3}{2}(t^2 - 4t - 8) + \frac{23}{2}e^{-\frac{1}{2}t}$

$= 3(2t - 4) + \frac{23}{2}e^{-\frac{1}{2}t}$ ✓

Lineare ODE (symbolisches Lösungsverfahren)

Normalform : $y'(t) + p(t) \cdot y(t) = s(t)$

Rezept erstellen ① & ②

$y'(t) + p(t) \cdot y(t) = s(t) \quad | \cdot \mu(t) \neq 0$

$\mu(t)y'(t) + \mu(t)p(t)y(t) = s(t)\mu(t)$

PR $\mu'(t)$ => ① $\mu(t) = e^{\int p(t)\,dt}$

... ② $y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)s(t)\,dt + C\right]$

Bsp. ODE $\begin{cases} 2y'(t) + y(t) = 3t^2 \\ y(0) = 1 \end{cases}$ IC

Normalform : $y'(t) + \frac{1}{2}y(t) = \frac{3}{2}t^2$

① $\mu(t) = e^{\int \frac{1}{2}dt} = e^{\frac{1}{2}t}$

② $\int \mu(t) \cdot s(t)\,dt = \frac{3}{2}\int t^2 e^{\frac{1}{2}t}\,dt$

$= 3(t^2 - 4t + 8)e^{\frac{1}{2}t}$

Allgemeine Lösung : $y(t) = 3(t^2 - 4t + 8) + C \cdot e^{-\frac{1}{2}t}$

$C \Rightarrow y(0) = 1 : 3 \cdot 8 + C = 1 \quad C = -23$

Spezielle Lösung => $y(t) = 3(t^2 - 4t + 8) - e^{\frac{23}{t/2}}$

Differentialgleichungen

Separierbare ODE (sog. Leibniz'sches Lösungsverfahren)

Normalform: $M(t) + N(y(t)) \cdot y'(t) = 0$

① $H_1(t) = \int M(t)\,dt$, $H_2(y) = \int N(y)\,dy$

② $H_1(t) + H_2(y(t)) = C \quad \Big| \frac{d}{dt}$

$M(t) + N(y(t)) \cdot y'(t) = 0$

mit Hauptsatz & Kettenregel

Bsp. ODE: $y'(t) = \dfrac{1+3t^2}{3y^2(t) - 6y(t)}$

Normalform: $\underbrace{-(1+3t^2)}_{M(t)} + \underbrace{(3y^2(t) - 6y(t))}_{N(y(t))} \cdot y'(t) = 0$

① $H_1(t) = -\int (1+3t^2)\,dt = -t - t^3$

$H_2(y) = \int (3y^2 - 6y)\,dy = y^3 - 3y^2$

② $\underline{-t - t^3 + y^3(t) - 3y^2(t) = C}$

Bifurkations-Diagramm

Bsp. SERIE 8 Aufgabe 6

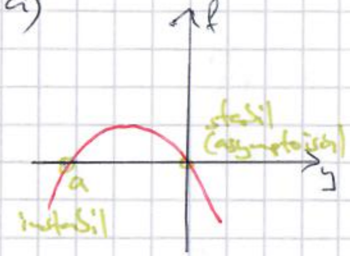
$\frac{d}{dt} y(t) = y(a-y)$

a) drei Fälle
$a<0$, $a=0$, $a>0$
- kritische Punkte
- Stabilität & Gleichgewicht

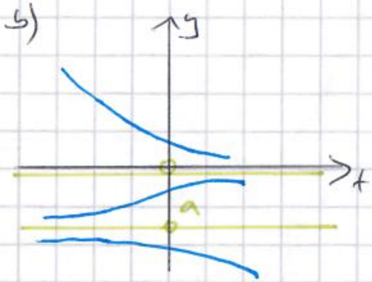
b) representative Schar

Fall 1: $a<0$, NS: $y_1 = a$, $y_2 = 0$

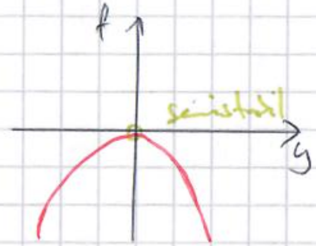
a)

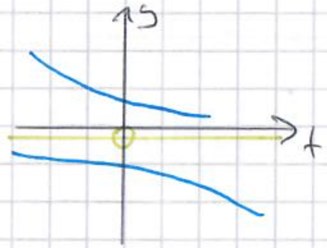


b)

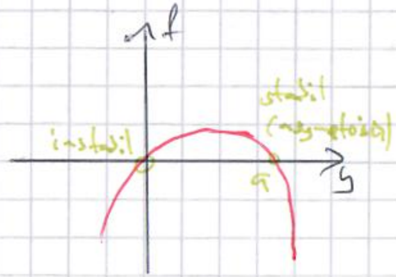


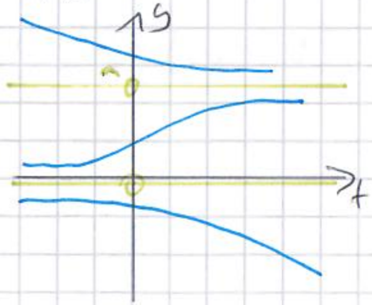
Fall 2: $a = 0$, $f(y) = -y^2$, NS: $y_1 = y_2 = 0$





Fall 3: $a>0$, NS: $y_1 = 0$, $y_2 = a$





WICHTIG

e^x Ableitung

$f(x) = e^x$ | $f(x) = e^{3x+3}$
$f'(x) = e^x$ | $f'(x) = 3 \cdot e^{3x+3}$

ln Ableitung

$f(x) = \ln(x)$ | $f(x) = \ln(x^4 + 2x^2)$
$f'(x) = \frac{1}{x}$ | $f'(x) = \frac{1}{x^4+2x^2} \cdot 4x^3 + 4x$

$\ln(3x) = \ln(3) + \ln(x)$

Logarithmus

$\log_{10}(100) = 2 \Rightarrow 10^2 = 100$

Regeln:

1. $\log_2(100) = \frac{\log_{10}(100)}{\log_{10}(2)}$
2. $\log_2(5x) = \log_2(5) + \log_2(x)$
3. $\log_2(x^7) = 7 \cdot \log_2(x)$
4. $\log_2(\sqrt[3]{7}) = \frac{\log_2(7)}{3}$
5. $\log_2(\frac{2}{x}) = \log_2(2) - \log_2(x)$

Additionstheorem

$2\sin(x) \cdot \cos(x) = \sin(2x)$