MRT Summary

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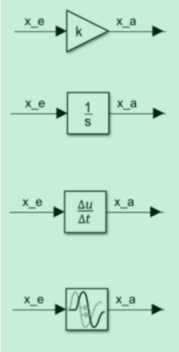
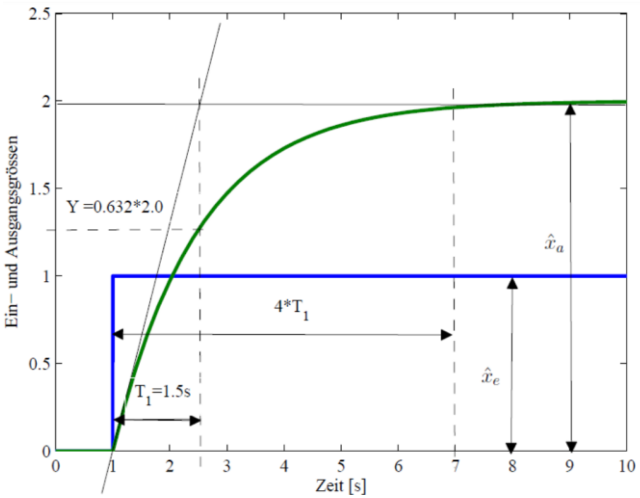
	Mathematical representation	Simulink representation
Proportional element (P-gain)	$x_a(t) = k \cdot x_e(t)$	
Integral element (I-gain)	$x_a(t) = \int x_e(t)dt$	
Differential element (D-gain)	$x_a(t) = \frac{dx_e(t)}{dt}$	
Dead-time element (T_t-gain)	$x_a(t) = x_e(t - T_t)$	



PT1 Systems

$$\dot{x}_a(t) = -\frac{1}{T_1} x_a(t) + \frac{K_s}{T_1} x_e(t)$$

State: $\dot{x}_a(t)$; Output: $x_a(t)$; Gain: K_s; Input: $x_e(t)$; Time constant: T_1



Input step function at steady state: $K_s = \frac{\Delta \widehat{x_a}}{\Delta \widehat{x_e}}$

Laplace

Operator: $s = \frac{d}{dt}$ Operator: $\frac{1}{s} = \int$

Initial value theorem

$$f(0) = \lim_{s \to +\infty} sF(s)$$

Final value theorem

$$\lim_{t \to +\infty} f(t) = \lim_{s \to 0} sF(s)$$

Impulse function $\mathcal{L}\{\delta(t)\} = 1$

Step function $\mathcal{L}\{\sigma(t)\} = \frac{1}{s}$

Sine function $\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2}$

Exponential function $\mathcal{L}\{e^{-\alpha t}\} = \frac{1}{s + \alpha}$

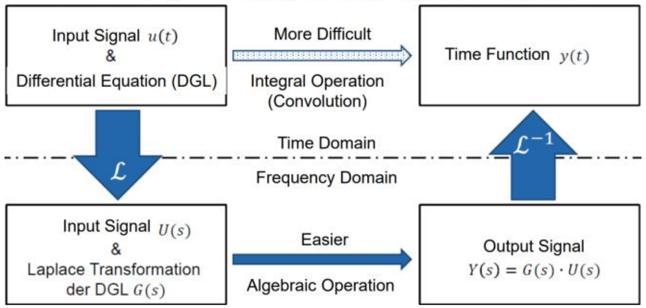
Further Laplace transforms: Papula p.358

Laplace inverse

- The inverse Laplace transformation **is the reverse transformation** from the image function to the time function.

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) e^{st} ds \qquad F(s) \bullet\!\!-\!\!\circ f(t)$$

1) The analysis of dynamic systems usually **starts with differential equations** which describe the behaviour of a control system.
2) The original differential equations are **transformed into image functions**, then integral and differential calculations can be performed as algebraic calculations.
3) When **the final result** in the image function is found, it **can be transformed back** to time domain (original form) **using the inverse Laplace transformation.**



Transfer function

For PT1 Systems: $G(s) = \frac{K_s}{T_1 s + 1}$

For PT2 Systems: Mass damper

Second Order DGL (Mechanical System) $m\ddot{x}(t) + b\dot{x}(t) + kx(t) = kf(t)$

Laplace Transformation $ms^2X(s) + bsX(s) + kX(s) = kF(s)$

Solution $X(s) = \frac{k}{ms^2 + bs + k} F(s) = G(s)F(s)$

Transfer Function $\frac{X(s)}{F(s)} = G(s)$

where $G(s) = \frac{k}{ms^2 + bs + k}$

Generally for PT2:

$$(a_2 s^2 + a_1 s + a_0) X_a(s) = (b_1 s + b_0) X_e(s)$$

$$G(s) = \frac{X_a(s)}{X_e(s)} = \frac{b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}$$

$$G(s) = \frac{\text{Laplace transformation of output signal} \quad X_a(s)}{\text{Laplace transformation of input signal} \quad X_e(s)}$$

Time constant $T_{vp} = -\frac{1}{p_v}$

(no particular meaning) $T_{vz} = -\frac{1}{z_v}$

Static Gain $K_S = \frac{b_0}{a_0}$

PT2 Parameters

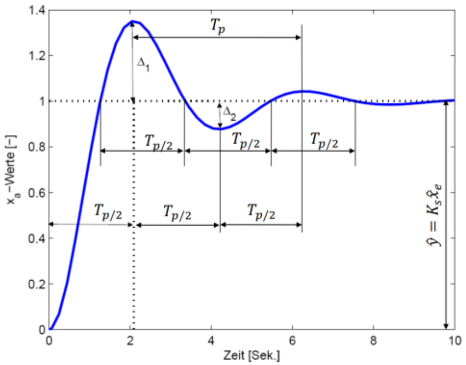
$0 \le \xi < 1$ $\quad$ $x_a(t) = K_s \hat{x}_e \left(1 - \frac{\omega_n}{\omega_d} e^{-\xi\omega_n t} \cdot \sin(\omega_d t + \Phi)\right)$ $\quad$ $\Phi = \cos^{-1}\xi$ $\quad$ $\frac{\omega_n}{\omega_d} = \frac{1}{\sqrt{1-\xi^2}}$

Natural Frequency ω_n $\quad$ $\omega_n = \frac{\omega_d}{\sqrt{1-\xi^2}}$

Damped Natural Frequency ω_d $\quad$ $\omega_d = \omega_n\sqrt{1-\xi^2}$

Damping Ratio ξ $\quad$ $\xi = \frac{\vartheta}{\sqrt{\pi^2 + \vartheta^2}}$

Logarithmic decrement ϑ $\quad$ $\vartheta = \frac{\pi\xi}{\sqrt{1-\xi^2}}$



$\omega_d = \frac{2\pi}{T_p}$

$\vartheta = \ln\frac{\Delta_1}{\Delta_2} = \frac{1}{n}\ln\frac{\Delta_k}{\Delta_{k+n}}$

$\xi = \frac{\vartheta}{\sqrt{\pi^2+\vartheta^2}}$

$\omega_n = \frac{\omega_d}{\sqrt{1-\xi^2}}$

first extreme value Δ_1 (first overshoot) at $t_1 = T_p/2$ $\quad$ $\Delta_1 = \hat{y}e^{-\vartheta}$ $\quad$ $t_1 = \frac{\pi}{\omega_d}$

second extreme value Δ_2 (first undershoot) at $t_2 = 2T_p/2$ $\quad$ $\Delta_2 = \hat{y}e^{-2\vartheta}$ $\quad$ $t_2 = \frac{2\pi}{\omega_d}$

k-th extreme value Δ_k (Over or undershoot) at $t_k = k2T_p/2$ $\quad$ $\Delta_k = \hat{y}e^{-k\vartheta}$ $\quad$ $t_k = \frac{k\pi}{\omega_d}$

Logarithmic Decrement $\quad$ $\vartheta = -\frac{1}{k}\ln\left(\frac{\Delta_k}{\hat{y}}\right)$ $\quad$ $\vartheta = \frac{1}{n}\ln\frac{\Delta_k}{\Delta_{k+n}}$

Time constant T_1 $\quad$ $T_1 = \frac{2\xi}{\omega_n}$

for $0 \le \xi < 1$

Time constant T_2 $\quad$ $T_2 = \frac{1}{\omega_n}$

TF Representations

Time-const. representation: $G(s) = \frac{K_S}{(sT_1+1)(sT_2+1)}$

Polynomial repr.: $G(s) = \frac{K_S{\omega_n}^2}{s^2 + 2\xi\omega_n s + {\omega_n}^2}$

Pole-zero repr.: $G(s) = \frac{K_S{\omega_n}^2}{(s-p_1)(s-p_2)}$

Polynomial representation

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0}$$

Pole-Zero representation

$$G(s) = k\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \quad \text{mit } k = \frac{b_m}{a_n}$$

Time constant representation

$$G(s) = K_S\frac{(T_{1z}s+1)(T_{2z}s+1)\cdots(T_{mz}s+1)}{(T_{1p}s+1)(T_{2p}s+1)\cdots(T_{np}s+1)} \quad K_S = \frac{b_0}{a_0} \quad T_{vp} = -\frac{1}{p_v} \quad T_{vz} = -\frac{1}{z_v}$$

Summation representation

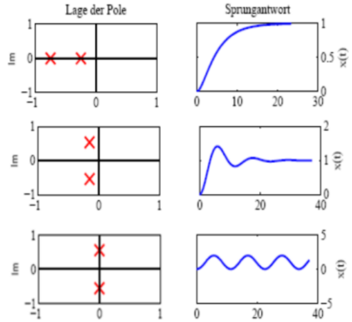
$$G(s) = \underbrace{k}_{P} + \underbrace{\frac{r_i}{s}}_{I} + \underbrace{\frac{r_k}{s-p_k}}_{PT1} + \dots + \underbrace{\frac{r_l}{(s-p_l)^n}}_{PTn} + \dots + \underbrace{\frac{a_m s + b_m}{s^2 + 2\xi_m\omega_{n_m}s + \omega_{n_m}^2}}_{PT2\ schwingfähig} + \dots$$

Poles and stability

$$p_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

Case1	$\xi > 1$	Two real poles	$p_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$
Case2	$\xi = 1$	Doble real poles	$p_{1,2} = -\xi\omega_n$
Case3	$\xi < 1$	Complex conjugate poles	$p_{1,2} = -\xi\omega_n \pm j\omega_d$
Case4	$\xi = 0$	Double imaginary poles	$p_{1,2} = \pm j\omega_n$

Poles located in the **left-hand** side → System **STABLE** of the complex plain $\quad$ $\omega_n > 0$, $\xi \ge 0$

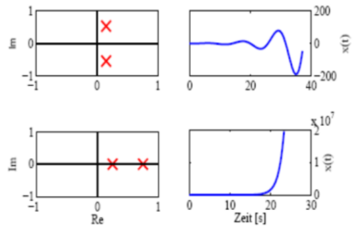


$\xi \ge 1$

$0 < \xi < 1$

$\xi = 0$

Poles located in the **right-hand** side → System **UNSTABLE** of the complex plain $\quad$ $\omega_n > 0$, $\xi < 0$



$-1 < \xi < 0$

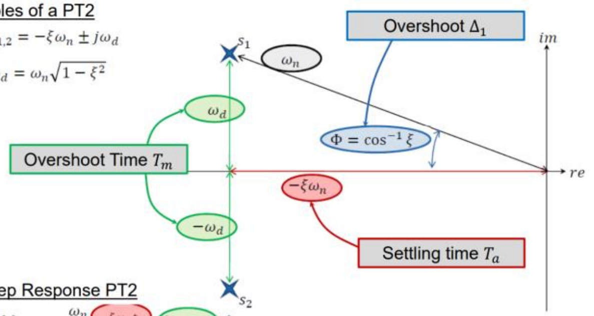
$\xi \le -1$

Conjugated Poles

Poles of a PT2

$p_{1,2} = -\xi\omega_n \pm j\omega_d$

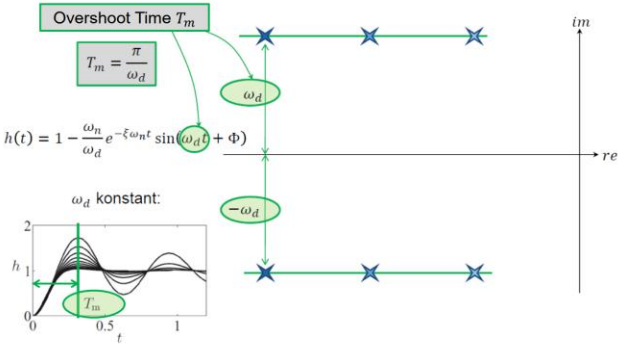
$\omega_d = \omega_n\sqrt{1-\xi^2}$



Step Response PT2

$h(t) = 1 - \frac{\omega_n}{\omega_d} e^{-\xi\omega_n t} \sin(\omega_d t + \Phi)$

$p_{1,2} = -\xi\omega_n \pm j\omega_d$



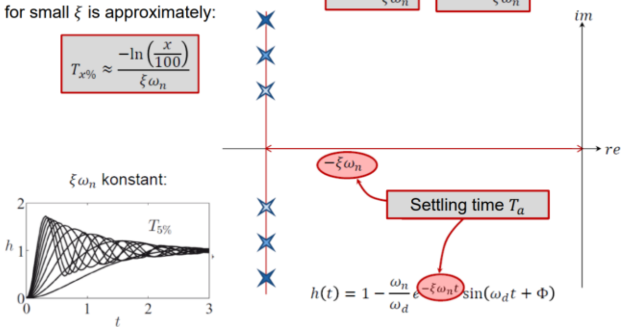
$p_{1,2} = -\xi\omega_n \pm j\omega_d$

Time within a tolerance band $\pm x\%$

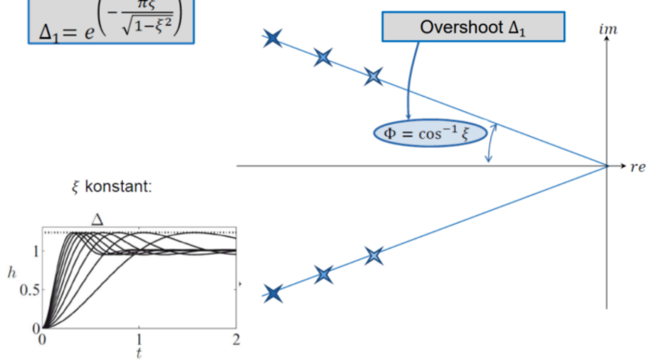
$T_{5\%} \approx \frac{3}{\xi\omega_n}$ $\quad T_{2\%} \approx \frac{4}{\xi\omega_n}$

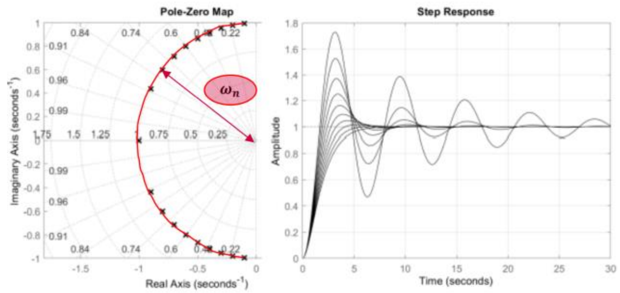
for small ξ is approximately:

$T_{x\%} \approx \frac{-\ln\left(\frac{x}{100}\right)}{\xi\omega_n}$



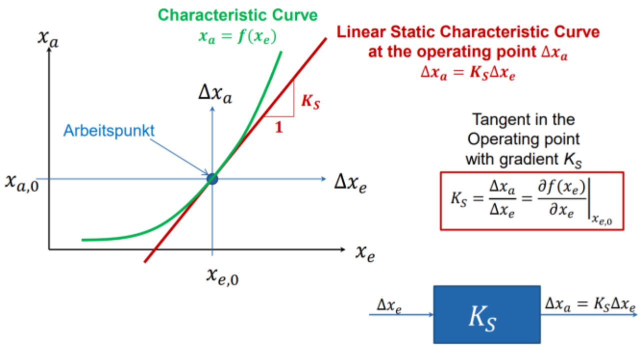
$$\Delta_1 = e^{\left(-\frac{\pi\xi}{\sqrt{1-\xi^2}}\right)}$$





Linear Systems

Static Gain



Transfer functions

The Laplace-Transformation of a step function with amplitude $\hat{x}_e$ is: $X_e(s) = \frac{\hat{x}_e}{s}$

Thus, the Laplace of the output is: $X_a(s) = G(s)X_e(s) = G(s)\frac{\hat{x}_e}{s}$

According to the final value theorem of Laplace

$$\lim_{t\to\infty} X_a(t) = \lim_{s\to 0} sX_a(s) = \lim_{s\to 0} sG(s)\frac{\hat{x}_e}{s} = \lim_{s\to 0} G(s)\hat{x}_e = G(0)\hat{x}_e = K_s\hat{x}_e$$

For a transfer function given by: $G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \dots + a_0}$

Static gain: $K_s = \lim_{s\to 0} G(s) = G(0) = \frac{b_0}{a_0}$

Proportional behaviour (P-element):

$a_0 \neq 0$ und $b_0 \neq 0$: $\quad K_s = \frac{b_0}{a_0}$

Integral behaviour (I-element):

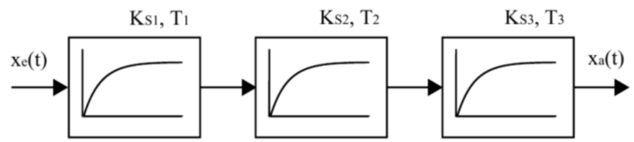
$a_0 = 0$ und $b_0 \neq 0$: $\quad K_s = \infty$

Differential behaviour (D-element):

$a_0 \neq 0$ und $b_0 = 0$: $\quad K_s = 0$

Series functions

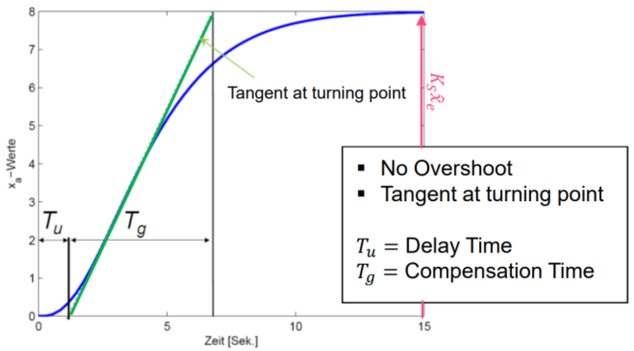
The series connection of "n" PT1 elements is called PTn element.
The next figure shows the block diagram for a PT3 element, each block contains one PT1 element.

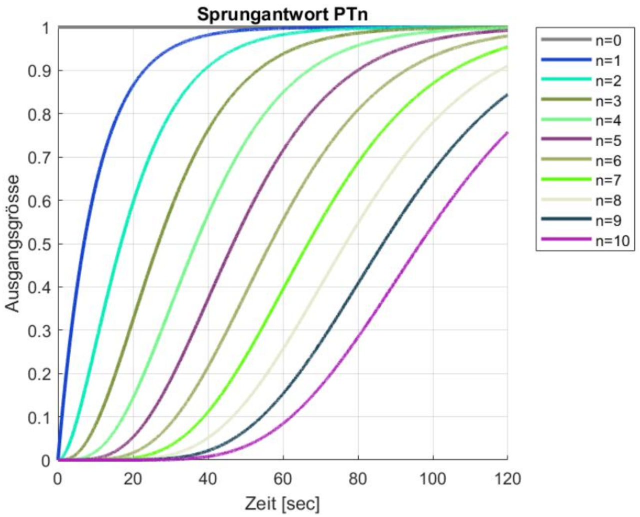


Each PT1-element has the same time constant, as result the new system has $T_m = \frac{1}{n}\sum_{i=1}^{n} T_i$ also known as **the mean time constant**.

$$G(s) = \frac{K_{S1}}{(T_1 s+1)} \cdot \frac{K_{S2}}{(T_2 s+1)} \cdot \frac{K_{S3}}{(T_3 s+1)} \cdot \dots = \frac{K_S}{(T_m s+1)^n}$$

Step response of PTn Systems





Mean time constant T_m

There are two methods to find T_m :

- Method of the tangent at turning point
- Method of the time-percentage characteristic value

Tangent at turning point

Reading data from the previous page:

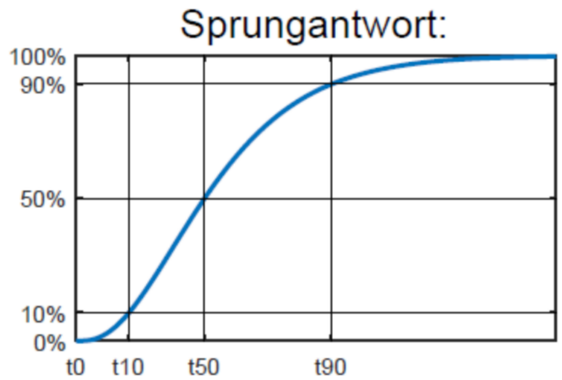
$$G(s) = \frac{K_S}{(T_m s + 1)^n}$$

n	$\frac{T_g}{T_u}$	$\frac{T_g}{T_m}$	$\frac{T_u}{T_m}$
2	9.71	2.72	0.28
3	4.61	3.69	0.80
4	3.14	4.46	1.42
5	2.44	5.12	2.10
6	2.03	5.70	2.81
7	1.75	6.23	3.55
8	1.56	6.71	4.30
9	1.41	7.16	5.08
10	1.29	7.59	5.87

Order	n
Static gain	K_S
Mean time-constant	T_m
Time Delay	T_u
Compensation Time	T_g

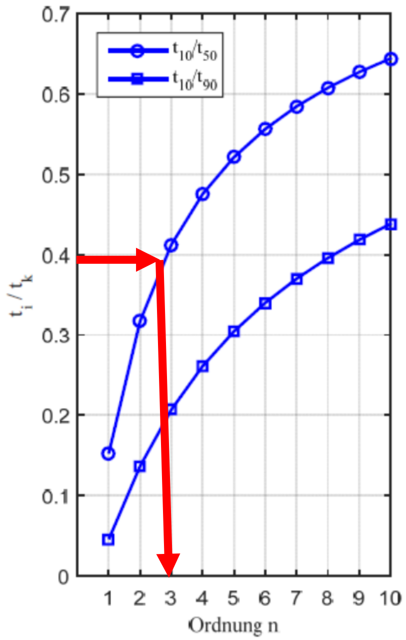
Time-percentage characteristic value

1. Break up the step response by the y-axis:

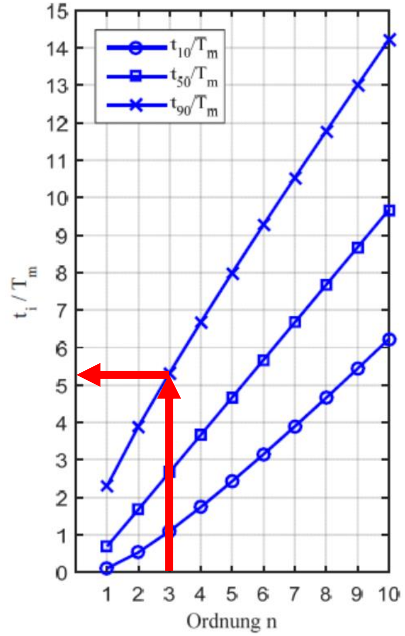


2. Find the time at the intersection of the percentage values: $\frac{t_{10}}{t_{50}} = \frac{t_i}{t_k}$ or $\frac{t_{10}}{t_{90}} = \frac{t_i}{t_k}$

3. Determine the order of the system:



4. Calculate the mean time constant by utilizing this plot: $T_M = value \cdot t_{10}$



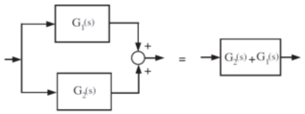
Block Diagram algebra

- Series connection

```
>> G=G1*G2
```

- Parallel connection

```
>> G=G1+G2
```



- Feedback

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>> G=G1/(1+G1*G2)
>> G=feedback(G1,G2)
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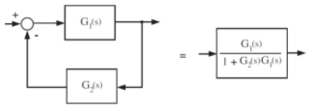
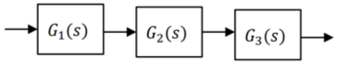


- Similar rules as in normal algebra can be used in blocks:

$$G(s) = G_1(s) * G_2(s) * G_3(s)$$



- If $G_2(s)$ is unknown and everything else is known, then:

$$G_2(s) = \frac{G(s)}{G_1(s) * G_3(s)}$$